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On the Ratios of the Individual Latent Roots to the Trace of a Wishart Matrix

A. W. DAVIS

C.S.I.R.O., Adelaide, Australia

A simple relationship is given between the exact null distribution $g_{m,n}^{(J)}$ of the J-th largest latent root of an $m \times m$ Wishart matrix on n degrees of freedom, and the distribution $f_{m,n}^{(J)}$ of the ratio of this root to the trace of the matrix. Explicit expressions for certain $f_{m,n}^{(J)}$ may thus be obtained from previous results for the corresponding $g_{m,n}^{(J)}$.

1. INTRODUCTION

In a recent paper, Krishnaiah and Waikar [5] have discussed tests of equality of the latent roots of certain matrices against various classes of alternatives. These tests are based on ratios of the latent roots of certain random matrices; in particular, on the ratios of the individual roots to the trace. In this note, we present a simple relationship between the marginal distribution of a latent root of a central Wishart matrix $\mathbf{S}$ and the distribution of the ratio of this root to the trace. Unfortunately, no corresponding results have yet been found for the matrices $\mathbf{S}_1\mathbf{S}_2^{-1}$ and $\mathbf{S}_1(\mathbf{S}_1 + \mathbf{S}_2)^{-1}$.

2. RATIOS OF THE INDIVIDUAL ROOTS TO THE TRACE

Let $\mathbf{S}$ be an $m \times m$ central Wishart matrix on n degrees of freedom, having latent roots $0 < l_m < \cdots < l_1 < \infty$, and let $u_i = l_i/\sum l_j$ $(i = 1,..., m; \sum_{i=1}^{m} u_i = 1)$. If $f_{m,n}^{(J)}$, $g_{m,n}^{(J)}$ denote the marginal densities of u_J, l_J respectively, and

$$\mathscr{L}(h(w)) = \int_0^\infty e^{-sw} h(w)\, dw$$

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denotes the Laplace transform, then we prove in the next section that

$$\mathscr{L}\left((1+w)^{\frac{1}{2}mn-2} f^{(J)}_{m,n}\left(\frac{1}{1+w}\right)\right) = 2\Gamma(\tfrac{1}{2}mn)\, e^{s} s^{-\frac{1}{2}mn+1} g^{(J)}_{m,n}(2s). \tag{2.1}$$

In principle, this result allows us to deduce the $f^{(J)}_{m,n}$ from expressions for the $g^{(J)}_{m,n}$ [6, 3]. In particular, if $p_{m;n_1,n_2}$ is the null density of trace $\mathbf{S}_1(\mathbf{S}_1 + \mathbf{S}_2)^{-1}$, where $\mathbf{S}_1$, $\mathbf{S}_2$ are independent $m \times m$ Wishart matrices on n_1, n_2 degrees of freedom, respectively, then we obtain for the largest ratio u_1

$$f^{(1)}_{m,n}(u) = k_1(m,n)\, u^{\frac{1}{2}mn-2} p_{m-1;n-1,m+2}\left(\frac{1-u}{u}\right), \tag{2.2}$$

where

$$k_1(m,n) = [\pi^{1/2}\Gamma(\tfrac{1}{2}mn)/\Gamma(\tfrac{1}{2}m)\Gamma(\tfrac{1}{2}n)] \prod_{i=0}^{m-2} [\Gamma(\tfrac{1}{2}(m+2-i))/\Gamma(\tfrac{1}{2}(n+m+1-i))]. \tag{2.3}$$

It follows from known results for p [2] that u_1 has range $(m^{-1}, 1)$, and that $f^{(1)}_{m,n}(u)$ is piecewise analytic in the intervals between the points $u = m^{-1}$, $(m-1)^{-1}, \ldots, 2^{-1}, 1$. As $n \to \infty$, $n^{1/2}(mu_1 - 1)$ tends to a limiting distribution, so that for very large n the distribution of u_1 is mainly concentrated in $(m^{-1}, (m-1)^{-1})$.

When $m = 2$,

$$\operatorname{Prob}(u_1 < u) = 1 - 2^{n-1}(u - u^2)^{\frac{1}{2}(n-1)}, \qquad (\tfrac{1}{2} < u < 1), \tag{2.4}$$

and $\frac{1}{2}n(2u_1 - 1)^2$ is asymptotically negative exponential with mean 1.

When $m = 3$, we obtain from [2, Eq. (3.4)] and (2.2)

$$\begin{aligned} f^{(1)}_{3,n}(u) &= k(n)\{2^n u^{n-2}(1-2u)^{\frac{1}{2}n} + u^{\frac{1}{2}n-2}[\tfrac{1}{2}n(1-u)^{n-2}(3u-1)^2 - (1-u)^n]\}, \\ &\qquad\qquad (\tfrac{1}{3} < u < \tfrac{1}{2}), \\ f^{(1)}_{3,n}(u) &= k(n)\, u^{\frac{1}{2}n-2}[\tfrac{1}{2}n(1-u)^{n-2}(3u-1)^2 - (1-u)^n], \qquad (\tfrac{1}{2} < u < 1), \end{aligned} \tag{2.5}$$

where $k(n) = 1/2\,B(\tfrac{1}{2}n + 1, n-1)$. The limiting density of $w = n^{1/2}(3u_1 - 1)$ in this case is

$$h(w) = \left(\frac{3}{2\pi}\right)^{1/2} [e^{-3w^2/2} + (\tfrac{9}{8}w^2 - 1)\, e^{-3w^2/8}], \qquad (0 < w < \infty), \tag{2.6}$$

with mean $3(3/2\pi)^{1/2} = 2.0730$, variance $(29\pi - 81)/6\pi = 0.5361$, and Pearson parameters $\beta_1^{1/2} = 0.4983$, $\beta_2 = 3.2071$. Approximate upper 5% and 1% points are 3.381 and 4.021, respectively.

Since the median root l_2 in the case $m = 3$ is a gamma variate with parameter $n - 1$, it follows from (2.1) that $2u_2$ is a beta variate with parameters $n - 1$, $\frac{1}{2}n + 1$.

For $m > 3$, the densities become increasingly complicated, with convolution-type terms. No further approximations have been found for the limiting distributions.

3. Proof of (2.1)

The joint nonnull density of $u_1, \ldots, u_{m-1}$ has been given in [4] and [5], and reduces in the null case (see [1]) to

$$\theta(u_1, \ldots, u_{m-1}) = k_2(m, n) \prod_{i=1}^{m} u_i^{\frac{1}{2}(n-m-1)} \prod_{i<j} (u_i - u_j),$$

where

$$k_2(m, n) = \pi^{\frac{1}{2}m} \Gamma(\tfrac{1}{2}mn) \Big/ \Big[\prod_{i=0}^{m-1} \Gamma(\tfrac{1}{2}(m - i))\, \Gamma(\tfrac{1}{2}(n - i)) \Big].$$

The u_i lie in the $(m - 1)$ dimensional region $\mathscr{E}_{m-1} = \{0 < u_m < \cdots < u_1 < 1, \sum_{i=1}^{m} u_i = 1\}$. Writing the left-hand side of (2.1) in the form

$$\int_{\mathscr{E}_{m-1}} \exp[-s(1 - u_J)/u_J]\, u_J^{-\frac{1}{2}mn} \theta(u_1, \ldots, u_{m-1}) \prod_{i=1}^{m-1} du_i, \tag{3.1}$$

we make the transformation

$$v_i = 2su_i/u_J \qquad (i = 1, \ldots, m;\ s > 0) \tag{3.2}$$

to new variables $v_1, \ldots, v_{J-1}, v_{J+1}, \ldots, v_m$. This maps $\mathscr{E}_{m-1}$ onto the region

$$\mathscr{F}_{m-1}(J, s) = \{0 < v_m < \cdots < v_{J+1} < v_J \equiv 2s < v_{J-1} < \cdots < v_1 < \infty\}. \tag{3.3}$$

Since

$$s(1 - u_J)/u_J = \tfrac{1}{2} \sum_{\substack{i=1 \\ i \neq J}}^{m} v_i, \tag{3.4}$$

and the transformation (3.2) has Jacobian

$$\partial(v_1, \ldots, v_{J-1}, v_{J+1}, \ldots, v_m)/\partial(u_1, \ldots, u_{m-1}) = (2s)^{m-1}/u_J^{m}, \tag{3.5}$$

the integral (3.1) becomes

$$2\Gamma(\tfrac{1}{2}mn)\, e^{s} s^{-\frac{1}{2}mn+1} \int_{\mathscr{F}_{m-1}(J,s)} \psi(v_1, \ldots, v_m) \prod_{\substack{i=1 \\ i \neq J}}^{m} dv_i , \tag{3.6}$$

where $\psi(v_1, \ldots, v_m)$ is the joint null density of $l_1, \ldots, l_m$. This is equivalent to the right-hand side of (2.1).

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